Attachment II: Proof of the Equivalence of Extensive Structures

Proof of the Equivalence of Extensive Structure

We now give the proof that the both extensive structures as introduced in Chapter 4, are equivalent. Firstly, we introduce some statements related to the weak order.

Weak Order

Lemma 1

Let (A, •≥) be a weak order. Then it holds for all a, b, c ∈ A:

i) a ·•> b, b •≥ c => a •> c
ii) a •≥ b, b•> c => a •> c.
♦

Proof:
i) Assume the statement is wrong. Then there exist a, b, c such that a •> b, b •≥ c and not (a •> c). Because of completeness not (a •> c) implies c •≥ a. Because of transitivity b •≥ c and c •≥ a imply b •≥ a. This is a contradiction to a •> b.

ii) Similar to i).
♦

Lemma 2

Let be (A, •≥, o) an extensive structure. Then it holds for all a,b,c ∈ A:

i) a ≈ b <=> a o c ≈ b o c
ii) a •> b <=> a o c •> b o c
♦
Proof:

Let be a ≈ b. Then a •≥ b and b •≥ a. Then because of monotonicity

a o c •≥ b o c and b o c •≥ a o c.

Hence a o c ≈ b o c.

Now assume a o c ≈ b o c. Then a o c •≥ b o c and b o c •≥ a o c, Because of monotonicity a •≥ b and b •≥ a. Then a ≈ b.

ii) The statement is equivalent to not (a •>b) <=> not(a o c •> b o c). Because of completeness this is equivalent to b •≥ a <=> b o c •≥ a o c. This statement is true because of monotonicity.
♦

A similar modification of the extensive structure was introduced by Bollmann /BOLL84/ for entropy structures. There it was shown that the axioms for the entropy structure are equivalent to the axioms of the modified extensive structure. Because the extensive structure is a special case of an entropy structure that proof can also be used to show the equivalence of the extensive structure and the modified extensive structure. We will give a slightly different proof here.

Theorem 1

Let **P** be a non-empty set, •≥ a binary relation on **P** and o a closed binary operation. Then (**P**, •≥ o) is an extensive structure iff it is a modified extensive structure.

Proof (Extensive Structure to Modified Extensive Structure):

Assume (**P**, •≥, o) is an extensive structure.

Weak order: trivial
Weak Associativity: trivial

Weak Commutativity: ∃ an additive function f with

$$f(a \text{ o } b) = f(a) + f(b) \text{ and } a \bullet\geq b \Leftrightarrow f(a) \geq f(b).$$

Then for all a, b ∈ **P**

f (a o b)	=	f (a) + f (b),
	=	f (b) + f (a),
	=	f (b o a),

and hence a o b ≈ b o a.

Weak monotonicity: Is weaker than monotonicity of the extensive structure.

Archimedean Axiom: Let be a •> b and c, d arbitrary. Then there exist k such that

$$(k\ a) \text{ o } c \bullet\geq (k\ b) \text{ o } d.$$

Then we obtain

$$((k + 1)\ a\ ) \text{ o } c \bullet\geq a \text{ o } (k\ b \text{ o } d),$$

because of monotonicity.

Because $a \bullet > b$ and lemma 2.ii we obtain $a(kb) \circ d > b(kb) \circ d$
and hence $((k+1)a) \circ c > ((k+1)b) \circ d$
q.e.d

Proof (Modified Extensive Structure to Extensive Structure):

Assume (**P**, $\bullet\geq$, o) is a modified extensive structure.

Weak order: trivial

Weak associativity: trivial

Monotonicity: Firstly, we have to show for all $a,b,c,d \in A$

$$a \bullet\geq b \Leftrightarrow a \circ c \bullet\geq b \circ c,$$
$$a \bullet\geq b \Rightarrow a \circ c \bullet\geq b \circ c.$$

hold because of weak monotonicity. We have still to show

$$a \circ c \bullet\geq b \circ c \Rightarrow a \bullet\geq b.$$

Assume the statement is false Then there exist $a,b,c \in A$ with

$$a \circ c \bullet\geq b \circ c$$
and
$$b \bullet> a.$$

This implies
$$(n\,a) \circ c \bullet\geq (n\,b) \circ c,$$

for all $n \geq 1$. We prove this by induction.

For $n = 1$ it holds because of the assumption: $a \circ c \bullet\geq b \circ c$. We assume the statement is correct for n. It holds:

$$(n+1)\,a \circ c \approx a(na) \circ c \bullet\geq a(nb) \circ c \approx (nb)(a \circ c) \bullet\geq (nb)(b \circ c) \approx (n+1)b \circ c,$$

for all n.
♦

Because of the Archimedean axiom of the modified extensive structure there exists $k \geq 1$ such that
$$(k\,b) \circ c \bullet> (k\,a) \circ c.$$

This is a contradiction. Hence we obtain

$$a \bullet\geq b \Leftrightarrow a \circ c \bullet\geq b \circ c.$$

Because of weak commutativity we also have

$$a \circ c \bullet\!\geq b \circ c \Leftrightarrow c \circ a \bullet\!\geq c \circ b.$$

Archimedean Axiom: The Archimedean axiom of the extensive structure is weaker than the Archimedean axiom of the modified extensive structure.